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Нахождение решения одной смешанной задачи для дифференциального уравнения четвертого порядка с производными в граничных условиях

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Аннотация. В статье рассматривается смешанная задача для обыкновенного дифференциального уравнения четвертого порядка с общими краевыми условиями. Решение задачи найдено вычетным методом. По схеме этого метода смешанная задача разбивается на две вспомогательные – спектральную задачу и задачу Коши. После исследования этих двух задач решение рассматриваемой смешанной задачи найдено в виде вычетного ряда. Показано, что решение рассматриваемой смешанной задачи охватывает не только параболические уравнения по Шилову, но и более широкие классы уравнений.

Ключевые слова: собственные значения, функция Грина, характеристический определитель, спектральная задача, формула разложения.

Introduction

As known from there are many processes connecting with process of heat conductivity and the diffusion movement are brought to the solution of Cauchy or mixed problems for parabolic equations [1–4].The study of the diffusion movement and heat conductivity process is of great importance in the mechanics of liquids and gases [5–7]. Thus, the study of the heat transfer process when laying rods of the same length with different heat transfer coefficients is expressed by differential equations in partial derivatives of the fourth order [8–10]. These equations usually reduce to equations of parabolic type in the sense of Petrovski [11–13]. There are more general equations of parabolic type than equations of parabolic type in the sense of Petrovski, for example, equations of parabolic type in the sense of Shilov [14–17]. Note, that some problems in quantum mechanics reduced to parabolic type equations in the sense of Shilov.

Statement of the problem

Consider the following problem:

$$\frac{\partial u(x,t)}{\partial t} = i\frac{\partial^4 u(x,t)}{\partial x^4} + q(x)\frac{\partial^2 u(x,t)}{\partial x^2},\ 0 < x < 1,\ t > 0 \tag{1}$$

$$u(x,0) = \varphi(x), \tag{2}$$

$$L_p(u) \equiv \sum_{k=1}^{3}\left(\alpha_{pk}\frac{\partial^{k-1}u(0,t)}{\partial^{k-1}x} + \beta_{pk}\frac{\partial^{k-1}u(1,t)}{\partial^{k-1}x}\right) = 0,\quad p = \overline{1,4}, \tag{3}$$

where α_{pk}, β_{pk} ($p = \overline{1,4}$, $k = \overline{1,3}$) are complex numbers, $q(x)$ and $\varphi(x)$ are complexvalued functions.

After application integral transformation

$$y(x,\lambda)=\int_0^\infty u(x,t)e^{-\lambda^4 t}dt$$

to the problem (1)–(3), we'll get following spectral problem:

$$iy^{IV}+q(x)y''-\lambda^4 y=-\varphi(x),\, 0<x<1 \tag{4}$$

$$L_p(y)\equiv\sum_{k=1}^{3}\left(\alpha_{pk}y^{(k-1)}(0,\lambda)+\beta_{pk}y^{(k-1)}(1,\lambda)\right)=0,\quad p=\overline{1,4}\,. \tag{5}$$

Roots of the characteristic equation in the sense of Birkhof corresponding to the equation (4) are found as follows [18]:

$$\theta_1=e^{-\frac{\pi}{8}i},\ \theta_2=i\,\theta,\ \theta_3=-\theta_1,\ \theta_4=-i\,\theta_1.$$

To find asymptotic of fundamental solutions of the equation (4) let's divide a complex plane λ into eight sectors by the following way [16]:

$$S_k=\left\{\lambda:-\lambda_1\,tg\frac{\pi}{8}<(-1)^{k-1}\lambda_2<\lambda_1 tg\frac{\pi}{8}\right\},\ k=1,2,$$

$$S_k=\left\{\lambda:\lambda_1\,tg\frac{\pi}{8}<(-1)^{k-1}\lambda_2<\lambda_1 tg\frac{3\pi}{8}\right\},\ k=3,4,$$

$$S_k=\left\{\lambda:\lambda_1\,tg\frac{3\pi}{8}<(-1)^{k-1}\lambda_2<\lambda_1 tg\frac{5\pi}{8}\right\},\ k=5,6,$$

$$S_k=\left\{\lambda:\lambda_1\,tg\frac{5\pi}{8}<(-1)^{k-1}\lambda_2<\lambda_1 tg\frac{7\pi}{8}\right\},\ k=7,8.$$

At $q(x)\in C^1[0,1]$ in the each sectors $S_k\left(k=\overline{1,8}\right)$ at large values of $|\lambda|$ the asymptotics of fundamental solution of the equation (4) have the following representation [18]:

$$\frac{d^m y_n(x,\lambda)}{dx^m}=(\lambda\theta_n)^m\left[1+\frac{1}{4\theta_n}\int_0^x q(\tau)d\tau+O\left(\frac{1}{\lambda^2}\right)\right]e^{\lambda\theta_n x}\,, \tag{6}$$

$$|\lambda|\to+\infty,\ \lambda\in S_p\left(p=\overline{1,8}\right),\ n=\overline{1,4},\ m=\overline{0,3}.$$

Green function of the spectral problem (4), (5) has the form [16]:

$$G(x,\xi,\lambda)=\frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)};\lambda\in S_p,\ p=\overline{1,8}. \tag{7}$$

$\Delta(\lambda)$ is called a characteristic determinant and is found as follows

$$\Delta(\lambda)=\begin{vmatrix} L_1(y_1) & L_1(y_2) & L_1(y_3) & L_1(y_4)\\ L_2(y_1) & L_2(y_2) & L_2(y_3) & L_2(y_4)\\ L_3(y_1) & L_3(y_2) & L_3(y_3) & L_3(y_4)\\ L_4(y_1) & L_4(y_2) & L_4(y_3) & L_4(y_4)\end{vmatrix} \tag{8}$$

and auxiliary determinant $\Delta(x,\xi,\lambda)$ is found as follows

$$\Delta(x,\xi,\lambda)=\begin{vmatrix} g(x,\xi,\lambda) & y_1(x,\lambda) & y_2(x,\lambda) & y_3(x,\lambda) & y_4(x,\lambda) \\ L_1(g)_x & L_1(y_1) & L_1(y_2) & L_1(y_3) & L_1(y_4) \\ L_2(g)_x & L_2(y_1) & L_2(y_2) & L_2(y_3) & L_2(y_4) \\ L_3(g)_x & L_3(y_1) & L_3(y_2) & L_3(y_3) & L_3(y_4) \\ L_4(g)_x & L_4(y_1) & L_4(y_2) & L_4(y_3) & L_4(y_4) \end{vmatrix}, \qquad (9)$$

where Cauchy function $g(x,\xi,\lambda)$ is found as follows [16]

$$g(x,\xi,\lambda)=\pm\frac{1}{2}\sum_{k=1}^{4} z_k(\xi,\lambda)y_k(x,\lambda)$$

"+" if $0\le\xi\le x\le 1$, "–" if $0\le x\le\xi\le 1$,

here $z_k(\xi,\lambda)=\dfrac{V_{4k}(\xi,\lambda)}{V(\xi,\lambda)}$, $k=\overline{1,4}$,

$V_{4k}(\xi,\lambda)$ is an algebraic complement of the fourth row element of Wronskian $V(\xi,\lambda)$.

To find the asymptotic of eigen values of spectral problem (4), (5) let's introduce the following notations:

$$L\left(\gamma_{k_1}^1\gamma_{k_2}^2\gamma_{k_3}^3\gamma_{k_4}^4\right)=\begin{vmatrix} \gamma_{1k_1}^1 & \gamma_{1k_2}^2 & \gamma_{1k_3}^3 & \gamma_{1k_4}^4 \\ \gamma_{2k_1}^1 & \gamma_{2k_2}^2 & \gamma_{2k_3}^3 & \gamma_{2k_4}^4 \\ \gamma_{3k_1}^1 & \gamma_{3k_2}^2 & \gamma_{3k_3}^3 & \gamma_{3k_4}^4 \\ \gamma_{4k_1}^1 & \gamma_{4k_2}^2 & \gamma_{4k_3}^3 & \gamma_{4k_4}^4 \end{vmatrix},$$

$$A_0=2L(\alpha_2\alpha_3\beta_2\beta_3),$$

$$B_0=2\left(L(\alpha_2\alpha_3\beta_1\beta_3)-L(\alpha_1\alpha_3\beta_2\beta_3)\right),$$

$$C_0=2\left(L(\alpha_1\alpha_2\alpha_3\beta_3)+L(\alpha_3\beta_1\beta_2\beta_3)\right),$$

$$g_k(x)=\frac{i}{4\theta_k}\int_0^x q(\tau)d\tau,\ k=\overline{1,4}.$$

Now to find asymptotic of eigenvalues of spectral problem (4), (5) consider the following theorem:

Theorem 1. Suppose, that α_{pk},β_{pk} ($p=\overline{1,4}$; $k=\overline{1,3}$) are complex numbers and $q(x)$ is complex-valued function, which has first order continuous derivatives on $[0,1]$. Then the zeros of the characteristic determinant $\Delta(\lambda)$ are countable set, single limit points of which is $\lambda=\infty$ and the following formulas for the asymptotic zeros are true:

$$\lambda_n^4=\left(4n^4+8n^3+6n^2\right)\pi^4 i-\pi^2n^2\left(\int_0^1 q(\tau)d\tau+4i\frac{B_0\pm C_0}{A_0}\right)+O(n)\quad n\to\pm\infty. \qquad (10)$$

Proof

Based on the property of determinant, the $\Delta(\lambda)$, found by formula (8) can be rewritten as follows:

$$\Delta(\lambda)=D_{12}(\lambda)e^{(\theta_1+\theta_2)\lambda}+D_{14}(\lambda)e^{(\theta_1+\theta_4)\lambda}+D_{23}(\lambda)e^{(\theta_2+\theta_3)\lambda}+D_{34}(\lambda)e^{(\theta_3+\theta_4)\lambda}+$$
$$+\sum_{k=1}^{4}D_k(\lambda)e^{\theta_k(\lambda)}+D_0(\lambda). \tag{11}$$

To find the main part of determinant $\Delta(\lambda)$ let's use the traditional method, that is equate the real part of exponents in pairs and selecting the straight lines or semi-straights, we'll get [18]:

$$\lambda_2=\lambda_1 tg\left(\frac{\pi}{8}+\frac{\pi}{4}(k-1)\right),\ k=\overline{1,8},\quad |\lambda_1|>R.$$

Choose those of the semi-strips, constructed from the above-mentioned semi-straights, where the main part of $\Delta(\lambda)$ hasan infinite number of zeroes. Let's denote as $\Pi_k(\lambda)$ and $\Delta_k(\lambda)$ $(k=\overline{1,4})$ these kinds of semi-strips and defined there the main parts of the determinant $\Delta(\lambda)$, correspondingly. Thus, the semi-strips $\Pi_k(\lambda)$ and $\Delta_k(\lambda)$ – the main part of $\Delta(\lambda)$ are defined as follows:

Consider the main part of $\Delta(\lambda)$ in the first quarter [19]:

$$\Pi_1(\lambda)=\left\{\lambda=\lambda_1+i\lambda_2:-\delta<\lambda_2-\lambda_1 tg\frac{\pi}{8}<\delta,\ \lambda_1>R\right\},$$

$$\Delta_1(\lambda)=\theta_1^9\lambda^{10}\left[\left(i\theta_1A_0\left(1+(1-i)g_1(1)\frac{1}{\lambda}\right)+(1+i)B_0\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right)e^{(\theta_1+\theta_2)\lambda}+\right.$$
$$\left.+\left(i\theta_1A_0\left(1+(1+i)g_1(1)\frac{1}{\lambda}\right)+(-1+i)B_0\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right)e^{(\theta_1+\theta_4)\lambda}+\left(2iC_0+O\left(\frac{1}{\lambda^2}\right)\right)e^{\theta_1\lambda}\right].$$

In the second quarter the main part has the form:

$$\Pi_2(\lambda)=\left\{\lambda=\lambda_1+i\lambda_2:-\delta<\lambda_2-\lambda_1 tg\frac{5\pi}{8}<\delta,\ \lambda_1>R\right\}$$

$$\Delta_2(\lambda)=\theta_1^9\lambda^{10}\left[\left(i\theta_1A_0\left(1+(1-i)g_1(1)\frac{1}{\lambda}\right)+(1+i)B_0\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right)e^{(\theta_1+\theta_2)\lambda}+\right.$$
$$\left.+\left(i\theta_1A_0\left(1+(-1-i)g_1(1)\frac{1}{\lambda}\right)+(1-i)B_0\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right)e^{(\theta_2+\theta_3)\lambda}+\left(2iC_0\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right)e^{\theta_2\lambda}\right].$$

The main part in the third quarter has the form

$$\Pi_3(\lambda)=\left\{\lambda=\lambda_1+i\lambda_2:-\delta<\lambda_2-\lambda_1 tg\frac{\pi}{8}<\delta,\ \lambda_1<-R\right\}$$

$$\Delta_3(\lambda)=\theta_1^9\lambda^{10}\left[\left(i\theta_1A_0\left(1+(-1-i)g_1(1)\frac{1}{\lambda}\right)+(1-i)B_0\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right)e^{(\theta_2+\theta_3)\lambda}+\right.$$
$$\left.+\left(i\theta_1A_0\left(1+(-1+i)g_1(1)\frac{1}{\lambda}\right)+(-1-i)B_0\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right)e^{(\theta_3+\theta_4)\lambda}+\left(-2iC_0\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right)e^{\theta_3\lambda}\right]$$

$$\Pi_4(\lambda)=\left\{\lambda=\lambda_1+i\lambda_2:-\delta<\lambda_2-\lambda_1 tg\frac{5\pi}{8}<\delta,\ \lambda_1<-R\right\}$$

$$\Delta_4(\lambda)=\theta_1^9\lambda^{10}\left[\left(i\theta_1A_0\left(1+(1+i)g_1(1)\frac{1}{\lambda}\right)+(-1+i)B_0\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right)e^{(\theta_1+\theta_4)\lambda}+\right.$$

$$\left.+\left(i\theta_1A_0\left(1+(-1+i)g_1(1)\frac{1}{\lambda}\right)+(-1-i)B_0\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right)e^{(\theta_3+\theta_4)\lambda}+\left(-2C_0\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right)\right)e^{\theta_4\lambda}\right],$$

Here $\delta>0$ and R is sufficiently large number.

Firstly, solve equation $\Delta_1(\lambda)=0$. For that introduce following notations:

$$\Delta_{11}(\lambda)=\left(i\theta_1A_0e^{\theta_2\lambda}+i\theta_1A_0e^{\theta_4\lambda}\right)\theta_1^9\lambda^{10}e^{\theta_1\lambda},$$

$$\Delta_{10}(\lambda)=\Delta_1(\lambda)-\Delta_{11}(\lambda)=\theta_1^9\lambda^{10}\left(i\theta_1A_0(1-i)g_1(1)+(1+i)B_0\frac{1}{\lambda}+O\left(\frac{1}{\lambda}\right)\right)e^{(\theta_1+\theta_2)\lambda}+$$

$$+\left(i\theta_1A_0(1+i)g_1(1)+(-1+i)B_0+O\left(\frac{1}{\lambda}\right)\right)e^{(\theta_1+\theta_4)\lambda}+\left(2iC_0+O\left(\frac{1}{\lambda}\right)\right)e^{\theta_1\lambda}\Bigg].$$

After solution of the equation $\Delta_{11}(\lambda)=0$ we'll get:

$$\mu_{1n}=\left(\frac{\pi}{2}+\pi n\right)\frac{1}{\theta_1},\ n\to+\infty.$$

To find roots of the equation $\Delta_1(\lambda)=0$ consider a following formulas [20]

$$\lambda_{1n}^m=\mu_{1n}^m+m\sum_{p=1}^{\infty}\frac{(-1)^p}{p}\mathop{res}_{\lambda=\mu_{1n}}\left[\lambda^{m-1}\left(\frac{\Delta_{10}(\lambda)}{\Delta_{11}(\lambda)}\right)^0\right].$$

In case of $m=4$ and $p=1$ we'll get

$$\lambda_{1n}^4=\mu_{1n}^4-4\mathop{res}_{\lambda=\mu_{1n}}\lambda^3\frac{\Delta_{10}(\lambda)}{\Delta_{11}(\lambda)}=\mu_{1n}^4-4\mathop{res}_{\lambda=\mu_{1n}}\lambda^2\times\left[\left(i\theta_1A_0(1-i)g_1(1)+(1+i)B_0+O\left(\frac{1}{\lambda}\right)\right)e^{\theta_2\lambda}+\right.$$

$$\left.+\left(i\theta_1A_0(1+i)g_1(1)+(-1+i)B_0+O\left(\frac{1}{\lambda}\right)\right)e^{\theta_4\lambda}+\left(2iC_0+O\left(\frac{1}{\lambda}\right)\right)\right]/\left(i\theta_1A_0e^{\theta_2\lambda}+i\theta_1A_0e^{\theta_4\lambda}\right).$$

It is easy to check, that μ_{1n} are simple poles of the function $\Delta_{11}(\lambda)$. According to that, we'll get

$$\lambda_{1n}^4=\mu_{1n}^4-4\mu_{1n}^2\frac{\theta_1A_0g_1(1)\left((1-i)e^{\theta_2\mu_{1n}}+(-1+i)e^{\theta_4\mu_{1n}}\right)e^{\theta_2\lambda}+B_0\left((1+i)e^{\theta_2\mu_{1n}}+(-1+i)e^{\theta_4\mu_{1n}}\right)+2iC_0}{i\theta_1\theta_2A_0\left(e^{\theta_2\mu_{1n}}-e^{\theta_4\mu_{1n}}\right)},$$

$$\lambda_{1n}^4=\mu_{1n}^4-4\mu_{1n}^2\left[\frac{g_1(1)}{i\theta_2}+\frac{B_0}{i\theta_1\theta_2A_0}-\frac{2C_0}{\theta_1\theta_2A_0\left(e^{\theta_2\mu_{1n}}-e^{-\theta_4\mu_{1n}}\right)}\right],$$

$$\lambda_{1n}^4=\mu_{1n}^4-4\mu_{1n}^2\left[\frac{g_1(1)}{i\theta_2}+\frac{B_0}{i\theta_1\theta_2A_0}-\frac{C_0}{\theta_1\theta_2A_0\sin\left(\frac{\pi}{2}+\pi n\right)}\right],$$

$$\lambda_{1n}^4=\mu_{1n}^4+4\mu_{1n}^2\left[\frac{g_1(1)}{\theta_1}+\frac{B_0}{\theta_1^2A_0}+\frac{C_0}{\theta_1^2A_0}(-1)^{(n-1)}\right].$$

Taking into account μ_{1n} and $g_1(1)$ into the last equality, we'll get:

$$\lambda_{1n}^4=\left(4n^4+8n^3+6n^2\right)\pi^4 i-\pi^4n^2\left(\int_0^1 q(\tau)d\tau+4i\frac{B_0+(-1)^{n-1}C_0}{A_0}\right)+O(n),\ n\to+\infty.$$

After solution the equation $\Delta_k(\lambda)=0\ (k=2,3,4)$ by the same way we'll get following asymptotic formulas:

$$\lambda_{2n}^4=\mu_{2n}^4-4\left[\frac{g_1(1)}{\theta_1}+\frac{B_0+(-1)^{n-1}C_0}{\theta_1^2A_0}\right]\mu_{2n}^2,$$

$$\mu_{2n}=\frac{(1+2n)\pi i}{2\theta_1},n\to-\infty,$$

$$\lambda_{3n}^4=\mu_{3n}^4+4\left[\frac{g_1(1)}{\theta_1}+\frac{B_0+(-1)^{n}C_0}{\theta_1^2A_0}\right]\mu_{3n}^2,$$

$$\mu_{3n}=\frac{(1+2n)\pi}{2\theta_1},n\to-\infty,$$

$$\lambda_{4n}^4=\mu_{4n}^4-4\left[\frac{g_1(1)}{\theta_1}+\frac{B_0+(-1)^{n}C_0}{\theta_1^2A_0}\right]\mu_{4n}^2,$$

$$\mu_{4n}=\frac{(1+2n)\pi i}{2\theta_1},\ n\to+\infty.$$

Substituting $\mu_{kn}\ (k=2,3,4)$ into equalities for $\lambda_{kn}^4\ (k=2,3,4)$ we'll get formula (10). The theorem is proved.

As it is known, that at $\operatorname{Re}q(x)>0, 0\le x\le 1$ equation (1) is parabolic in the sense of Shilov [21]. A following theorem allows us to find solution of the mixed problem (1)–(3) not only in case of parabolic in the sense of Shilov, but also wider classes:

Theorem 2. Suppose, that functions $g(x)$ and $\varphi(x)$ are satisfies to a following conditions $q(x)\in C^1[0,1], \varphi(x)\in C^2[0,1]$, $\varphi(0)=\varphi(1)=\varphi'(0)=\varphi'(1)=0$, $\operatorname{Re}q(x)>0$.If $A_0\ne 0$, coefficients of the boundary conditions are complex numbers and $\operatorname{Im}\frac{B_0\pm C_0}{A_0}$, then mixed problem (1)–(3) has following solution

$$u(x,t)=-i\sum_{k=1}^{4}\sum_{n=1}^{\infty}\underset{\lambda=\lambda_{km}}{res}\,\lambda^3e^{\lambda^4t}\int_0^1 G(x,\xi,\lambda)\varphi(\xi)d\xi, \qquad (12)$$

here $G(x,\xi,\lambda)$ is a Green function of the corresponding spectral problem, λ_{kn} $\left(k=\overline{1,4}; n=1,2,3,...\right)$ are all zeroes of the characteristic determinant $\Delta(\lambda)$.

Proof. Let's search solution of the mixed problem (1)–(3) as follows

$$u(x,t)=\sum_{k=1}^{4}\sum_{n=1}^{\infty}\underset{\lambda=\lambda_{km}}{res}\,\lambda^3\int_0^1 G(x,\xi,\lambda)z(\xi,t,\lambda)d\xi. \qquad (13)$$

Taking into account (12) into (1) and (2) we can find function $z(\xi,t,\lambda)$ in such form

$$z(\xi,t,\lambda) = -ie^{\lambda^4 t}\varphi(\xi).$$

Taking into account the last into (13), we'll get formula (12).

From condition $A_0 \neq 0$ can be said, that problem (4), (5) is regular [16; 18]. It means, that out of $\delta > 0$ neighborhood of zeros of the characteristic determinant $\Delta(\lambda)$ inequality

$$\left|G^{(k)}(x,\xi,\lambda)\right| \leq \frac{M_k(x,\xi,\lambda)}{|\lambda|^{3+k}},\ k = \overline{0,3},\ \lambda \in S_p\ (p = \overline{1,8}) \tag{14}$$

is true, where $M_k(x,\xi,\lambda)$ are positive, bounded with respect to x and ξ functions and analytic function with respect to λ-complex parameter. At the same time, under condition $A_0 \neq 0$ and $q(x) \in C^1[0,1]$, $\varphi(x) \in C^2[0,1]$, $\varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0$ for the function $\varphi(x)$ following formula of decomposition is true [16]:

$$\varphi(x) = -i\sum_{k=1}^{4}\sum_{n=1}^{\infty} \underset{\lambda=\lambda_{kn}}{res}\ \lambda^3 \int_0^1 G(x,\xi,\lambda)\varphi(\xi)d\xi. \tag{15}$$

Taking into account (15) we can see that series given by formula (12) satisfies to initial condition. As the Green function $G(x,\xi,\lambda)$ is a solution of the homogeneous equation, corresponding to (4), (5), the series (12) formally satisfies to the boundary condition (2).

It is necessary for (12) and series, obtaining by differentiating it four time with respect to x, and ones with respect to t toconvergence uniformly and absolutely. For that, taking into account conditions of the theorem and asymptotic of eigenvalues, defined by formula (10) we'll get:

$$\left|e^{t\lambda_{kn}^4}\right| = e^{t\,\mathrm{Re}\,\lambda_{kn}^4} = e^{-t\pi^2 n^2 \int_0^1 q(\tau)d\tau + O(n)}.$$

It shows, that if $\mathrm{Re}\int_0^1 q(\tau)d\tau > 0$ and formula satisfies according toWeierstrass theorem the functional series (12) uniformly and absolutely convergence. It means that our search formal operations are justified.

The theorem is proved.

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Finding of a Solution of One Mixed Problem for a Fourth-order Differential Equation with Derivatives in Boundary Conditions

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Abstract. Mixed problem for the fourth-order ordinary differential equation with general boundary conditions is considered in present paper. The solution of the problem is found by the deductible method. Following the scheme of this method, the mixed problem is divided into two auxiliary ones –spectral and the Cauchy problems. After studying these two problems, solution of the considered mixed problem is found by the deductive series. It is shown, that solution of considered mixed problem covers not only parabolic Shilov equations, but also wider classes of equations.

Keywords: eigenvalues, Green function, characteristic determinant, spectral problem, Formula of decomposition

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