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Уравнение диффузии излучения: явные аналитические решения с использованием усовершенствованного метода интегрального баланса[1]

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Разработаны приближенные явные аналитические решения уравнения диффузии излучения с применением двойного интегрирования в методе интегрального баланса. Этот метод позволяет разрабатывать приближенные решения закрытой формы. Решена проблема со ступенчатым изменением температуры поверхности и две задачи с зависящими от времени граничными условиями. Минимизация погрешности приближенных решений была проведена непосредственно путем минимизации остаточной функции определяющего уравнения.

Ключевые слова: *диффузия, метод интегрального баланса, приближенные решения, температура поверхности.*

1. Introduction

Heat wave behaviour of thermal diffusion due to radiation is a reasonable physical and mathematic interpretation of thermal energy transfer in a variety of applied problem related to astrophysical phenomena [1], plasma physics [2, 3], building insulation [4], etc. In general, this physical model assumes a diffusion approximation relating the local thermal flux at any point of the medium by the local gradient of the radiation energy density that is an approach known from the classical Fourier law. Following Smith [5] the one-dimensional energy transfer by radiation and absence of fluid motion is

$$\rho \frac{\partial e}{\partial t} = -\frac{\partial \Phi}{\partial x}, \ \Phi = -\frac{4}{3K\rho}\frac{\partial \sigma T^4}{\partial x} \tag{1}$$

The presentation of the thermal flux density Φ as a gradient of the 4[th] power of the local temperature is an accordance with the Rosseland approximation [1, 5, 6] which is valid for thick, non- opaque media in absence of fluid motion [5–9].

The diffusion equation (1) was solved for the first time by Barenblatt [10] by a self-similar solution and then refined by Zeldovich and Raizer [11]. As commented by Smith [5] these early attempts, especially the report of Hammer and Rosen [14] repeated the idea of the Barenblatt but met problems in defining the shape of the spatial temperature distribution profile. At the same time as the Barenblatt solution appeared the study of Marshak [9] was carried out (performed in 1944–1945 in Los Alamos Laboratory but published only in 1958 as it is especially mentioned in the publication) . The Marshak solution also tried to express the spatial temperature distribution as a series of parabolic profiles. We will especially consider this model in the article since the new method used applies a generalized parabolic profile.

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For further deep reading on the solutions related to radiation-diffusion equation we refer to [5–8] and the references therein.

The aim of the present work is to present an approximate closed form analytical solution allowing to estimate the penetration depth of the heat wave and the spatial temperature distribution applying an improved integral-balance approach [12, 13] already successfully applied to nonlinear heat conduction problems modelled by degenerate parabolic equations (see further in the text).

2. The nonlinear heat radiation diffusion equation

In accordance with Smith [5] the internal energy e and Rosseland mean opacity K are approximated by power-law of temperature T and density ρ as

$$e = f\frac{T^{\beta}}{\rho^{\mu}},\ \frac{1}{K} = g\frac{T^{\alpha}}{\rho^{\lambda}} \tag{2}$$

where $f = 3.4\,MJ/kg$ and $g = 1/7200\,g/cm^2$ are dimensional constants (especially for the case of gold) [14].

The exponents α and β are positive constants and in accordance with [5] and [14] we have $\alpha = 1.5$ and $\beta = 1.6$. With the power-laws (2) the energy balance (1) yields the following diffusion equation

$$\frac{\partial T^{\beta}}{\partial t} = D\frac{\partial^2 T^{4+\alpha}}{\partial x^2},\ D = \frac{16\varepsilon}{\beta}\frac{g\sigma}{3f\rho^{2-\mu+\lambda}},\ \varepsilon = \frac{\beta}{4+\alpha} \tag{3}$$

It noteworthy that the dimension of the diffusion coefficient D in (3) is $m^2/sK^{[4+\alpha-\beta]}$ and only in the case when $\alpha = 0$ and $\beta = 1$ we have $D \equiv \left[m^2/sK^3\right]$.

Smith [5] used the parameter $U = T^{4+\alpha-\beta}$ to transform (3) into a wave equation

$$\frac{\partial U}{\partial t} = D\left[\frac{1}{1-\varepsilon}\left(\frac{\partial U}{\partial x}\right)^2 + \frac{U}{\varepsilon}\frac{\partial^2 U}{\partial x^2}\right] \tag{4}$$

Equation (4) excludes the case $\varepsilon \neq 1$ corresponding to the linear diffusion model. The transformed equation (4) was solved by Smith [5] with a series approximation

$$U(z,t) = \sum_{i=1}^{\infty} c_i(t) z^i,\ z = x_F - x \tag{5}$$

In the context of the Smith solution x_F defines the front of a heat wave beyond which the medium is undisturbed, i. e. $T = 0$ for $x \geq x_F$ that is $U(0,t) = 0$. To this point, we will stop considering the Smith solution but will refer to it in comments on results developed in this article.

3. Solution approach

3.1. Integral balance approach in brief

The integral-balance method used in this work is based on the concept that the diffusant (heat or mass) penetrates the undisturbed medium at a final depth $\delta(t)$ which evolves in time. Therefore, the common boundary conditions at infinity ($T(\infty) = 0$ and $\partial T(\infty,t)/\partial x = 0$) can be replaced by

$$T(\delta) = 0 \text{ and } \frac{\partial T}{\partial x}(\delta) = 0 \tag{6}$$

The conditions (6) define a sharp-front movement $\delta(t)$ of the boundary between disturbed and undisturbed medium. The position $\delta(t)$ is unknown and should be determined trough the solution. When the classical heat diffusion problem is at issue and the thermal diffusivity is temperature-independent (i. e. $a = a_0 = const.$) [12, 13, 15]

$$\frac{\partial T(x,t)}{\partial t} = a_0 \frac{\partial^2 T}{\partial x^2} \tag{7}$$

The integration of eq. (7) over a finite penetration depth δ and applying the Leibniz rule for differentiation under the integral sign results in (8)

$$\frac{d}{dt}\int_0^{\delta} T(x,t)\,dx = -a_0 \frac{\partial T}{\partial x}(0,t) \tag{8}$$

or equivalently as

$$\int_0^{x} \frac{\partial T(x,t)}{\partial t}dx + \int_x^{\delta} \frac{\partial T(x,t)}{\partial t}dx = -a_0 \frac{\partial T}{\partial x}(0,t) \tag{9}$$

Physically, eq.(8), as well as (9) imply that the total thermal energy accumulated into the finite layer (from $x = 0$ to $x = \delta$) is balanced by the heat flux at the interface $x = 0$. Equations (8) and (9) are the principle relationships of the simplest version of the integral-balance method known as *Heat-balance Integral Method* (**HBIM**) [16]. After this first step, replacing T by an assumed profile T_a (expressed as a function of the relative space coordinate x/δ), then the integration in (8) results in an ordinary differential equation about $\delta(t)$ [12, 13, 16, 17]. The principle drawback of (8) is that the right-side depends on gradient expressed through the type of the assumed profile.

An improvement, avoiding the drawback of HBIM is the double integration method (**DIM**) [12, 13]. The first step of DIM is integration of from 0 to x, namely

$$\int_0^{x} \frac{\partial T(x,t)}{\partial t}dx = a_0 \frac{\partial T(x,t)}{\partial x} - a_0 \frac{\partial T(0,t)}{\partial x} \tag{10}$$

Subtracting (10) from (9) we get

$$\int_x^{\delta} \frac{\partial T(x,t)}{\partial t}dx = a_0 \frac{\partial T(x,t)}{\partial x} \tag{11}$$

Equation (11) has the same physical meaning as eq. (8). The integration of (11) from 0 to δ results in

$$\int_0^{\delta}\left(\int_x^{\delta} \frac{\partial T}{\partial t}dx\right)dx = a_0 T(0,t) \tag{12}$$

The integral relation (12) allows to work with either integer-order time-derivatives (as in the present case) or with time-fractional derivatives [19] where the Leibniz rule is inapplicable

If the thermal diffusivity is non-linear and expressed as a power-law $a = a_p T^m$ ($m > 0$), corresponding to degenerate diffusion problems) then equation (12) takes the forms [12, 13]

$$\int_0^{\delta}\left(\int_x^{\delta} \frac{\partial T}{\partial t}dx\right)dx = \frac{a_p}{m+1}\left[T(0,t)\right]^{m+1} \tag{13}$$

The integral relation (13) will be used further in this work in the solutions of the problems at issue.

3.2. Transformation of the governing equation and degenerate diffusion equation

Here, following the notations of Smith [5] in section 2 and denoting $\theta = T^{\beta}$ which leads to $T = \theta^{1/\beta}$ and $T^{4+\alpha} = \theta^{(4+\alpha)/\beta}$. Now, we can present (3) as

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta^w}{\partial x^2},\ w = \frac{4+\alpha}{\beta} \tag{14}$$

From (14) it is obvious that $\varepsilon = 1/w$ defined in (3). From the defined values of α and β we have that $w > 1$. Equation (14) can be presented as [12]

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D_w \theta^m \frac{\partial \theta}{\partial x}\right],\ \theta^m \frac{\partial \theta}{\partial x} = \frac{1}{m+1}\frac{\partial \theta^{m+1}}{\partial x} \tag{15}$$

where $D_w = wD$ and $m = w-1 > 0$

The transformed equation (15) allows direct application of the Double-Integral Method, as it is demonstrated next. This is a classical example of the so-called s*low diffusion models* [19, 20] which degenerates at $\theta = 0$, that is at the front of the moving solution. A change of variables $\varphi = \theta^m$ and $\tau = t/m$ allows eq. (15) to be expressed as

$$\frac{\partial \varphi}{\partial \tau} = D_w\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + m\varphi\frac{\partial^2 \varphi}{\partial x^2}\right] \tag{16}$$

Equation (16) has the same structure as (4) solved by Smith [5] and reveals a superposition of non-linear wave propagation and diffusion; it was successfully solved by HBIM in [15].

4. DIM solutions: constant density cases

4.1. Step change of temperature at the boundary

This example only demonstrates the technique of DIM applicable to the radiation diffusion equation and how the approximate profile should be determined. For the sake of simplicity a Dirichlet (a step change in the boundary condition at $x = 0$) is considered to equation (15).

The application of DIM to (15) yields the following integral relations (equivalent) to (13), namely

$$\frac{d}{dt}\int_0^{\delta}\int_x^{\delta}\theta(x,t)dxdx = D_w\frac{\theta^m(0,t)}{m+1}, \tag{17}$$

For Dirichlet boundary condition $T_s = const.$ we have $\theta_s = \theta(0,t) = T_s^{\beta} = const.$ The DIM solution assumes an approximate profile as parabolic one with unspecified exponent $\theta = \theta_s(1 - x/\delta)^n$ which satisfies the boundary conditions (6) for any positive value of the exponent n [12, 21]. Applying DIM we get

$$\delta_{D_w(DIM)} = \sqrt{D_w t}\sqrt{\frac{(n+1)(n+2)}{m+1}} \tag{18}$$

Then, the normalized approximate solution is [12].

$$\Theta = \frac{\theta}{\theta_s} = \left(1 - \frac{x}{\sqrt{D_w t}\sqrt{\frac{(n+1)(n+2)}{m+1}}} \right)^n \tag{19}$$

Thus defining the Boltzmann similarity variable $\eta_D = x/\sqrt{D_w t}$. With the inverse change of variables $T = \theta^{1/\beta}$ and $T_s = \theta_s^{1/\beta}$ we get

$$T_{norm} = \frac{T}{T_s} = \left(1 - \frac{x}{\sqrt{D_w t}\sqrt{\frac{(n+1)(n+2)}{m+1}}} \right)^{\frac{n}{\beta}} \tag{20}$$

As it was established in [12], the exponent n of the solution (19) follows the rule $n = 1/m$ and therefore, the profile (20) has an exponent $1/m\beta$. Consequently with $m = w - 1 = 4 - 1 = 3$, and $\beta = 1.6$ we have $1/m\beta \approx 0.208$ and the normalized solution is

$$T_{norm} = \left(1 - \frac{\eta_D}{\sqrt{\frac{1+2m}{m^2(m+1)}}} \right)^{\frac{1}{m\beta}} = \left(1 - \frac{\eta_D}{0.440} \right)^{0.208} \tag{21 a, b}$$

Hence, for $\eta_D = x/\sqrt{D_w t} = 0.440$ we have $T_{norm} = 0$ and this point defines the front of the heat wave.

Even though, HBIM and DIM are not self-similar method of solution, the benefits of their application is the definition of the dimensionless space coordinate x/δ allowing to determine straightforwardly the desired similarity variable without preliminarily scaling of the modelling equation. This approach will be demonstrated effectively in the examples solved in the next points.

4.2. Temperature-independent properties with time-dependent boundary condition (Marshak's problem)

4.2.1. Marshak's approach

Following Marshak [9], when the material density does not vary and the medium just heat up, the equation takes the form (in the original notations) resembling (14)

$$\frac{\partial T}{\partial t} = \frac{D_M}{(p+4)} \frac{\partial^2 T^{4+p}}{\partial x^2}, D_M = \frac{4cl_0}{3} \tag{22}$$

The pseudo-diffusion coefficient D_M has a dimension m^3/sK^{3+p} as it was demonstrated in eq. (3). Marshak [9] considered time-dependent boundary condition: $T_s = J_0 \exp(2\alpha_s t)$, $x = 0$, where J_0 is the initial surface temperature at $t = 0$ and α_s is a time constant. The solution was developed as a series $f(z) = \sum_{i=0}^{\infty} A(1 - z/z_0)^i$, with the ansatzes (23).

$$f(z)=A\left(1-\frac{z}{z_0}\right)^{\frac{1}{p+3}},\ A=\frac{1}{J_0}\frac{(p+3)}{2L_0}z_0,\ z_0=\frac{2L_0}{(p+3)}J_0^{\left(\frac{p+3}{2}\right)},\ L_0=\sqrt{\frac{cl_0}{3\alpha_s}}=2\sqrt{D_M} \tag{23}$$

Here z_0 defines the front of the thermal wave. Marshak detected that only one term is enough to assure approximation error of 1,2 %, if $p=3$. Hence, with $p=3$ assumed by Marshak we have $1/(p+3)=1/6\approx 0.166$, $A=3z_0/J_0L_0$, $z_0=J_0^{\,3}L_0/3$ and the normalized solution of Marshak is

$$\frac{f(z)}{A}=\frac{f(z)}{3z_0/J_0L_0}=\left(1-\frac{z}{J_0^{\,3}L_0/3}\right)^{0.166} \tag{24}$$

4.2.2. DIM solution

The application of DIM (13) to (22) with $T_a=T_s(1-x/\delta)^n$, with $T_s=T_s(t)$ yields

$$\frac{d\left[J_0\exp(2\alpha_s t)\delta^2\right]}{dt}=D_M N(n,p)\left[J_0\exp(2\alpha_s t)\right]^{4+p},\ N(n,p)=\frac{(n+1)(n+2)}{(p+4)} \tag{25}$$

From (25) with $\delta(t=0)=0$ we have

$$\delta=\sqrt{\frac{D_M}{\alpha_s}J_0^{3+p}e^{2\alpha_s t(3+p)}}\sqrt{\frac{N(n,p)}{2(4+p)}} \tag{26}$$

The product $D_M J_0^{3+p}/\alpha_s$ has a dimension $\left[m^2\right]$ and therefore the ratio $x/\sqrt{(D_M/\alpha_s)J_0^{3+p}}$ is dimensionless. Hence, the normalized approximate solution is

$$\frac{T_a(x,t)}{J_0e^{2\alpha_s t}}=\left(1-\frac{\eta_M}{\sqrt{N(n,p)/2}}\right)^n \tag{27}$$

$$\eta_M=\frac{x}{\sqrt{\frac{D_M}{\alpha_s}J_0^{3+p}e^{2\alpha_s t(3+p)}}} \tag{28}$$

This solution defines the similarity variable η_M (28), while the denominator of (27) defines numerically the penetration depth.

The residual function of (22) is a measure of the error of approximation when the approximate solution is used. Following the methodology used in [12, 13] we have

$$R=\left(\frac{x}{\delta}\right)\frac{1}{\delta}\frac{d\delta}{dt}n\left(1-\frac{x}{\delta}\right)^{n-1}-\frac{D_M 4n(4n-1)}{\delta^2}\left(1-\frac{x}{\delta}\right)^{4n-2} \tag{29}$$

If R should be zero at the interface $x=0$, satisfying eq. (22), then from (29) it follows that $4n-1=0\rightarrow n=1/4$. Otherwise, the requirement $R(0,t)>0$ should be satisfied if $n<1/4$. At the front of the thermal layer when $x\rightarrow\delta$, the last term of (29) tells us that the condition $4n-2>0\rightarrow n>1/2$ should be obeyed. However, the first condition established at $x=0$ is more important since it allows determining the flux at the boundary. Moreover R can be presented as $R=r(z,t)/\delta^2$ [12], where with $z=x/\delta$ we may present the expanding in time thermal layer as a fixed boundary domain since $0\le z\le 1$. Hence, $r(z,t)$ is

$$r(z,t) = z\frac{1}{2}\frac{d\delta^2}{dt}n(1-z)^{n-1} - D_M 4n(4n-1)(1-z)^{4n-2} \tag{30}$$

Then, the mean squared error of approximation over the thermal layer is δ (i. e. from $z=0$ to $z=1$) is

$$E(z,t) = \int_0^1 \left[R(z,t)\right]^2 dz = \frac{1}{\delta^4}\int_0^1 \left[r(z,t)\right]^2 dz \tag{31}$$

Since, $E(z,t)$ decays in time with a speed proportional to $1/\delta^4$ it is important to minimize it at the beginning of the diffusion process. Then, following Myers [22] and setting $t=0$ in all time-dependent terms we get

$$E(z,0) = \frac{\left[4n(4n-1)\right]^2}{8n-1} \tag{32}$$

Hence, we get a minimal error of approximation at $t=0$ for $n=1/4$ which coincides with the value established earlier at $x=0$. Moreover, from the denominator of $E(z,0)$ it follows that $n>1/8$, a condition satisfied by $n=1/4$. However, this exponent is not the exact one [22]. Moreover, as it was reported in [14] the profile $T \approx T_s(1-x/x_F)^{1/4}$ is the solution of the steady-state equation $\partial^2 T^4/\partial x^2 = 0$, with x_F is the optical depth to the heat front.

Now, we recall that eq. (22) ca be presented as

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(D_M T^q \frac{\partial T}{\partial x}\right),\ q = 3+p \tag{33}$$

This is a well-known degenerate diffusion equation [19–21]. As it was established in [12, 15] and mentioned in the preceding example, the exponent of the approximate profile can be approximated as $n \approx 1/q$. Then, with $n = 1/(p+3)$ we have $N(n,p)=2$ and $\sqrt{N(n,p)/2} = 1$. Then, the normalized approximate profile is

$$\frac{T_a(x,t)}{J_0 e^{-2\alpha_s t}} \approx (1-\eta_M)^{\frac{1}{p+3}} \tag{34}$$

Hence, $\eta_M = 1$ defines the front of the thermal layer, that in dimensional form is

$$x_\delta = \sqrt{\frac{D_M}{\alpha_s} J_0^{3+p} e^{2\alpha_s t(3+p)}} \tag{35}$$

The penetration depth z_0 in the solution of Marshak is defined through the surface temperature by $J_0^{(3+p)/2}$ and this confirms qualitatively the DIM solution where $\delta \equiv J_0^{(3+p)/2}$. In most of cases the value of p is stipulated through the scaling exponents α and β in the approximations such as (2). The Marshak exponent $n = 0.166$ satisfies the condition $1/8 < n_M = 0.166 < 1/4$. The exponent $n = 1/4$ corresponds to $n = 1/(p+3)$ for $p=1$ and this provides $\sqrt{N(n,p)} = 1$. Moreover, from (22) we get $T^{4+p} \approx T^5$. With $\beta = 1$ in (22) which is equivalent to (14) we have $w = (4+\alpha)/\beta \approx 5.5$ and a scaling $T^w \approx T^{5.5}$.

Therefore, due to different formulations of the similarity variables and different solution techniques we got different exponents of the approximating parabolic profile. In this context, the method determining the exponent through minimization of the means-squared error of approximation is reliable [12, 13, 22] and we accept $1/8 < n_{opt} < 1/4$ for the DIM solution. Numerical tests with $E(z,t)$ varying n in the above defined range provided $n_{opt} \approx 0.153$ with $E(z,t) \approx 0.00245$.

The functional relationship of the similarity variables η_M (28) and x_δ (35) but the numerical factors are different. In order to compare the DIM solution to this of Marshak we present them in a fixed boundary range $0 \le z \le 1$ with $z = x/\delta = x/x_\delta$. The plots on Fig. 1a reveal that the present DIM solution with the optimal exponent is quite close to the Marshak solution. The use of the limit values of exponent $n = 1/4$ and $n = 1/8$ leads to unacceptable pointwise errors when compared to the Marshak solution while with $n_{opt} = 0.153$ we may reach error less than 0.02 as it is illustrated by the plots in shown Fig. 1b. All solutions demonstrate large errors near the steep front where the gradient tends to be infinite, but we know that this is inherent for integral-balance solutions of degenerate parabolic equation [12, 13].

4.3. Temperature-dependent properties and time-dependent boundary power-law condition (Garnier's problem)

4.3.1. Garrnier approach

Garnier et al. [8] considered the same postulation of the problem as in [5] (see eq. (2) and eq. (3)) and by change of variable $\xi = T^{4+\alpha}$ transformed eq. (1) into

$$\frac{\partial \xi^{\varepsilon}}{\partial t} = D_G \frac{\partial^2 \xi}{\partial x^2}, D_G = \frac{16}{12+3\alpha}\frac{g\sigma}{\rho_0^{2-\mu+\lambda}}, \varepsilon = \frac{\beta}{4+\alpha} \qquad (36)$$

if the scaling laws (2) are valid with $\alpha = 1.5$, $\beta = 1.6$, $\lambda = 0.2$ and $\mu = 0.14$.

Following Garnier et al. [8] the solution developed requires $2\beta < 4+\alpha$, thus assuring $\varepsilon \in (0, 1/2)$. Garnier et al. [8] accepted the finite depth $\delta(t)$ concept of the heat wave propagation with the condition $\xi(\delta) = \partial\xi(\delta)/\partial x = 0$. With a power law surface temperature $T_s(t) = T_{s0}(t/t_s)^k$, where k is a given exponent and t_s is a time scale, the validity of the solution Garnier et al. [8] imposed the condition $-1/(4+\alpha-\beta) < k < 1/(4+\alpha-2\beta)$. This approach looked for a self similar solution with ansatzes where Γ is a dimensionless parameter that parameterizes the differential equation about ξ.

$$\xi(x,t) = \xi_s(t)\xi(x/\delta),\ \ \xi_s(t) = T_{s0}^{4+\alpha}(t/t_s)^{q_0},\ \ \delta = \delta_0(t/t_s)^{n_G},\ \ q_0 = \frac{4+\alpha}{k},\ \ \delta_0 = \sqrt{\Gamma D_G}\sqrt{T_{s0}^{4+\alpha}}$$

(37)

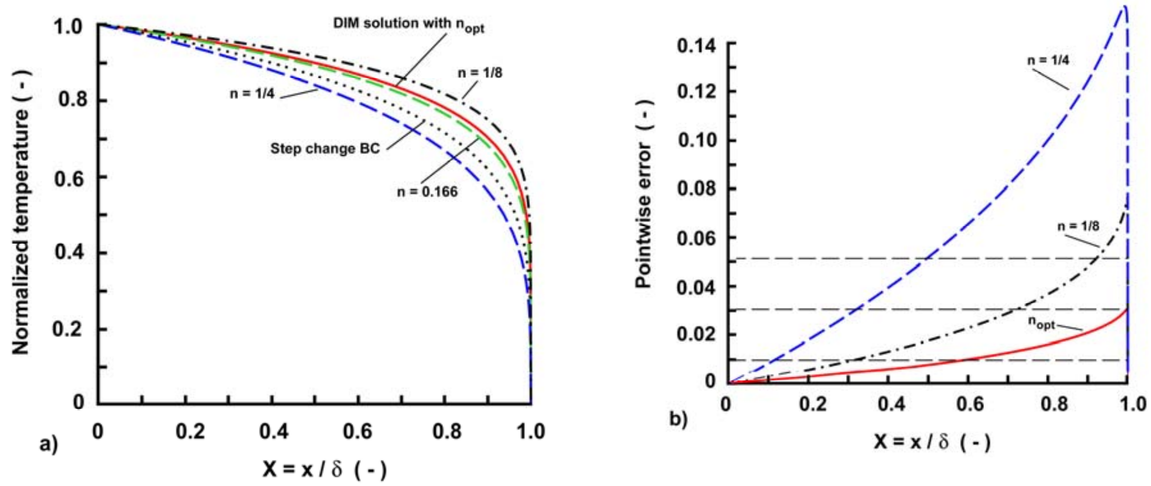

Fig. 1. Comparison of the approximate DIM solutions with the solution of Marshak. a) Normalized temperature profiles; b) Pointwise errors $\left|Y_{Marshak} - Y_{DIM}\right|$, where Y denote the normalized temperature (see eq. (27) and eq. (34)). **Note:** The dotted line presents the DIM solution of example 4,1 with a step change in the boundary condition. The horizontal dashed lines mark 3 levels of pointwise error: 0,01, 0,03 and 0,05

After substitution of (37) in (36) Garnier et al. [8] established that the exponent n_G should be $n_G = \left[1 + k\left(4 + \alpha - \beta\right)\right]/2$. Further, the solution was considered as an engeinvalue problem defining the eigenvalue-eigenfunction (Γ, ξ) depending only on the values of α, β and k. At this moment we stop the analysis of the solution of Garnier et al. [8] since it requires a numerical approach in contrast to the analytical DIM technique applied next.

4.3.2. DIM solution

The change of the variable as $\theta = \xi^{\varepsilon} = T^{4+\alpha}$ (as it was done in 3.2) transforms equation (36) as

$$\frac{\partial \theta}{\partial t} = D_G \frac{\partial^2 \theta^w}{\partial x^2}, \quad w = \frac{4+\alpha}{\beta} = \frac{1}{\varepsilon} \tag{38}$$

which is equivalent to eq. (14)

With an assumed profile $\theta = \theta_s \left(1 - x/\delta\right)^n$ and the DIM integral relationship (15) we get

$$\delta^2 = D_G \frac{N(n,w)}{kw+1} \frac{T_{s0}^{w-1}}{a_{ts}} t^{k(w-1)+1}, \quad N(n,w) = \frac{(n+1)(n+2)}{w}, \quad \left(t_s\right)^{kw-1} = a_{ts} = const. \tag{39}$$

Hence, the penetration depth is

$$\delta_H = \sqrt{D_G T_{s0}^{(w-1)} t^{n_H}} \sqrt{\frac{N(n,w)}{a_{ts}}} \tag{40}$$

$$n_H = k \frac{4+\alpha-\beta}{\beta} + 1 = k(w-1) + 1 \tag{41}$$

The DIM solution provides $\delta \equiv t^{n_H/2}$ and that $(n_H - 1) = (n_G - 1)/\beta$. With $\alpha = 1.5$ and $\beta = 1.6$ we have $\delta_H \equiv t^{1/2+1.218k}$, while $\delta_G \equiv t^{1/2+1.95k}$. For $k = 0$ both, δ_H and δ_G scale to $\sqrt{t}$.

Further, with the inverse transforms $T=\theta^{1/(4+\alpha)}$ the approximate solution $T_a(x,t)$ is

$$T_a(x,t)=T_{s0}\left(\frac{t}{t_s}\right)^k\left(1-\frac{x}{\delta_H}\right)^q,\ q=\frac{n}{4+\alpha} \tag{42}$$

The definition the residual function R_H of eq. (38), carried out in a manner already demonstrated in section 4.2.2, reveals that q should satisfy the condition $n>1/w$ and it is independent of k because the product $t^{-(w-1)k}\delta(d\delta/dt)$ emerging in expression of R_H is time-independent. Then, the restriction on the exponent in (42) is $q>\beta/(4+\alpha)$. If we suggest $q=\beta/(4+\alpha)$, then in (42) we should have $n=\beta>1$ which should provide an unphysical concave temperature profile of (39) as it was proved in [12, 13]. However, with the prescribed values of α and β we have $w\approx 3.437$ or $m=w-1=2.437$.

The optimization of the residual function with respect to q provides that for $m=2$ we have $q_{opt}\approx 0.666$, while for $m=3$ we have $q_{opt}\approx 0.640$. Now, from $q_{opt}=n_{opt}/(4+\alpha)$ we $n_{opt}(m=2)\approx 3.663$ and $n_{opt}(m=2)\approx 3.520$. Precisely, for $m=2.437$ we get $q_{opt(m=2.437)}\approx 0.636$ and consequently $n_{opt(m=2.437)}\approx 3.501$. Alternatively, if we suggest that the exponent q could be expressed $q_a(4+\alpha)$ it is easy to see that from the above established values about q_{opt} we have $q_{a(m=2)}\approx 0.666$ and $q_{a(m=3)}\approx 0.640$, while $q_{a(m=2.437)}\approx 0.649$.

With these estimates, the normalized temperature profile is

$$\frac{T_a(x,t)}{T_{s0}(t/t_s)^k}\approx\left(1-\frac{\eta_H}{\left[2.684/a_s(1+4.347k)\right]}\right)^{0.649} \tag{43}$$

where $\eta_H=x\big/\sqrt{D_G T_{s0}^{n_H/2}t^{n_H}}$ is the similarity variable.

The flux approximate profile follows directly from the definitions (1) and (36) and the solution (43) as

$$\Phi_a(x,t)=D_G\left[T_{s0}\left(\frac{t}{t_s}\right)^k\right]^4\frac{4q}{\delta_H}\left(1-\frac{x}{\delta_H}\right)^{4q-1} \tag{44}$$

Mean while the approximated surface flux $\Phi_a=-D_G\left[\partial T_a^4(0,t)/\partial x\right]$ is $\Phi_a=\left[T_{s0}(t/t_s)^k\right]^4(4q/\delta_H)$ or in normalized form as $\Phi_a\big/\left[T_{s0}(t/t_s)^k\right]^4=(4q/\delta_H)$. From the results (40) and (41) it follows that

$$\Phi_a\big/\left[T_{s0}(t/t_s)^k\right]^4\equiv q^4 t^{-\frac{1}{2}\left[k\frac{4+\alpha-\beta}{\beta}+1\right]}\equiv q^4 t^{-\frac{1}{2}+1.95k} \tag{45}$$

The normalized DIM solution DIM solution (43) as a function of the similarity variable n_H is shown in Fig. 2.

The condition of increasing surface flux when the surface temperature is prescribed as increasing power-law function of time follows from (45) and the requirement is $k<7\beta/(4+\alpha-\beta)\approx 2.871$. Hence with linear or quadratic ramp of surface temperature the surface flux will increase in time. Moreover, the normalized flux follows from (44) as

$$\frac{\Phi_a(x,t)}{D_G\left[T_{s0}(t/t_s)^k\right]^4 4q/\delta_H}=\left(1-\frac{x}{\delta_H}\right)^{4q-1} \tag{46}$$

This expression allows the present results to be compared at least qualitatively with the numerical solutions in [8]. For optimal $q_{a(m=2.437)} \approx 0.636$ we have $4q-1 \approx 1.544 > 1$. With such an exponent the parabolic profile (46) generates concave distributions (see Fig. 2) in contrast to temperature profiles which are convex.

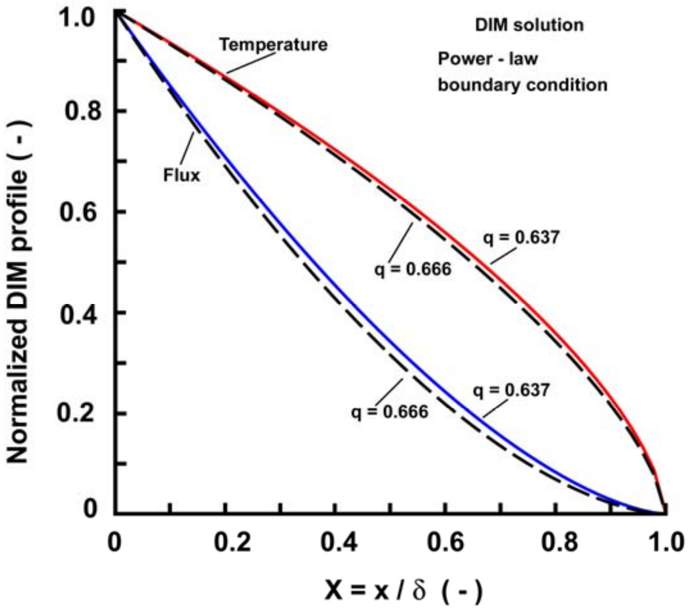

Fig. 2. DIM solutions as normalized profiles of both the temperature and the heat flux with power-law time-dependent boundary condition

The approximate DIM temperature profile and the results of Garnier et al. [8] (numerical solutions) are hard to be compared due to incompatibility of the methods used. However, we have to mention that the developed DIM solution confirms the numerical results of Garnier et al., especially the profile in Fig. 4 in [8], when the isothermal thermal shock condition were simulated. To be precise, all examples discussed here are within the limit of the isothermal thermal shock as in the first postulation of Marshak [9]. Other regimes discussed in [8] such as subsonic waves and the ablation problem are out of the scope of the present work.

3. Conclusions

This work demonstrated the efficiency of the double-integration version of the integral-balance method to solve the radiation diffusion equation. It was confirmed that this equation can be easily presented as a degenerate parabolic equation. Then, the technique developed in [12] was efficiently applied to find explicit approximate and closed form solutions. The main difference between the solutions of Marshak and Garnier commented here and the DIM solutions are that in the former studies ansatzes about the penetration depth are used, while DIM defines it as a step of the solution. The second important step in the present solution is the optimization procedure defining the optimal exponent of the parabolic profile [12]. This step, in fact, is application of the least squares method [23] since the entire function of the assumed parabolic profile is completely defined after determination of the penetration depth. Such procedure is not included in the previous solutions commented here.

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The radiation diffusion equation:
Explicit analytical solutions by improved integral-balance method

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Approximate explicit analytical solutions of the radiation diffusion equation by applying the double integration technique of the integral-balance method have been developed. The method allows approximate closed form solutions to be developed. A problem with a step change of the surface temperature and two problems with time-dependent boundary conditions have been solved. The error minimization of the approximate solutions has been developed straightforwardly by minimization of the residual function of the governing equation.

Keywords: *diffusion, integral-balance method, approximate solutions, surface temperature.*

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